6 OPTIMAL CONTROL

So far we were concerned with control design where the objective was either to stabilize a system (the regulator problem), or to track a reference input (the servomechanism problem). We can do better than this though! In particular, what if we wanted to design the "best" controller, where the word "best" is understood with respect to some measure of merit or performance index? In classical control design we have already seen the use of integral performance criteria (such as ITAE) in order to obtain desirable characteristic equations for use in pole placement. Other criteria could lead to minimizing the travel time (minimal time control), fuel consumption (minimal fuel control), miss distance (optimal randevouz), and so on. These requirements lead to the design of optimal controllers.

6.1 Optimal Control Problems

In general terms, the problem is to find a control law u for the system $\dot{x} = f(x, u)$ such that a certain index J is minimized. Therefore, the basic problem of optimal control is

$$\text{minimize } J = K(x_0, x_f) + \int_{t_0}^{t_f} L(x, u)\, dt\ ,$$

under the constraint

$$\dot{x} = f(x, u)\ .$$

K, L are specified functions, and

$$\left.\begin{array}{l} x_0(t_0) : \text{initial state (time)} \\ x_f(t_f) : \text{final state (time)} \end{array}\right\} \text{given or free}\ .$$

This formulation is general enough to allow for several interesting cases, for instance,

- $K = 0$, $L = 1 \Longrightarrow$ minimal time problem,
- $K = 0$, $L = |u| \Longrightarrow$ minimal fuel problem,

and so on.

Specifically, we have the following problem statement:

1. System equations $\dot{x} = f(x, u, t)$ where $x \in R^n$ is the state vector, and $u \in R^m$ is the controls vector.
2. Boundary conditions on the starting time, t_0, initial state $x_0 = x(t_0)$, final time t_f, and final state $x_f = x(t_f)$. These may or may not be given, therefore we can have a number of combinations fixed–free, free–free, free–fixed problems.

3. Performance index
$$J = K(x_f, t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)\, dt \; .$$
A few special cases for this are:
   - The Mayer problem,
   $$J = K(x_f, t_f) \; .$$
   - The Lagrange problem,
   $$J = \int_{t_0}^{t_f} L(x(t), u(t), t)\, dt \; .$$
   - The Bolza problem, both K and L are non–zero.
4. Constraints can be either on control; i.e., $|u_i| \leq 1$ (very common), or on the state; i.e., $G(x_f, t_f) = 0$ (target sets), $|x_i| \leq X_i$ (inequality constraints, very hard to handle in general). These constraints determine a set of admissible control histories, U, and a set of admissible state trajectories, X.

The general problem of optimal control can then be stated as:

> Find $u(\cdot) \in U$ which takes the system from x_0 at t_0 to x_f at t_f by $\dot{x} = f(x, u, t)$ in such a way at to minimize J while $x(\cdot) \in X$.

6.2 Examples

Some examples of optimal control problems are:

1. *Time Optimal Control*:
Consider $J = \int_{t_0}^{t_f} dt$ where t_0 is fixed and t_f is free. We can have fixed end points or belonging on target sets. Usually, we also need constraints on u to make the problem well–posed. As a particular example consider $\ddot{x} = u$, where $|u| \leq 1$. Say we start from initial conditions $x_0, \dot{x}_0$ both positive and we want to get to the origin $x_f = \dot{x}_f = 0$, as quickly as possible. We can see that since we initially have positive x and positive $\dot{x}$ we must apply full negative control $u = -1$ in order to get negative $\dot{x}$ (i.e., towards the origin) while x remains positive. Then at some instant we should switch to full positive control $u = +1$ to stop at $x = 0$ with zero speed. The precise instant of switching from $u = -1$ to $u = +1$ is, of course, not known for now. This is an example of a bang–bang control problem, which most time optimal control problems lead to.
2. *Fuel Optimal Control*:
A typical example is,
$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} |u_i|\, dt \; .$$

Typically, such problems lead to bang–bang controls and with t_f free, the problem may be ill posed for certain initial conditions — i.e., if no restrictions on t_f are placed minimum fuel could mean coast to the destination with very small speed.

3. *Mimimum Integral Square Error*:
   Here,
   $$J = \int_{t_0}^{t_f} x^T x \, dt \quad \text{or} \quad J = \int_{t_0}^{t_f} x^T Q x \, dt \ ,$$
   where Q is a symmetric and positive definite matrix. Typically we need constraints on u to prevent it from becoming infinitely large. In the special case of linear state feedback, we get the familiar ISE criterion.

4. *Mimimum Energy Problems*:
   Here,
   $$J = \int_{t_0}^{t_f} u^T R u \, dt \ ,$$
   where R is symmetric and positive definite.

5. *Final Value Optimal Control*:
   Here, $J = K(x_f, t_f)$, for example
   $$J = \sum_{i=1}^{n} (x_{if} - x_i(t_f))^2 \ .$$
   Combinations of the above are, of course, also possible examples.

6.3 Calculus of Variations

A real function of a real variable is a map between a real number to another real number. A map between a function to a real number is called a functional. The performance index J is an example of a functional. Minimization of a functional is the subject of a branch of mathematics, called calculus of variations. The simplest problem of the calculus of variations is,

$$\min J = \int_{t_0}^{t_f} L(x, \dot{x}, t) \, dt \ ,$$

where x is a scalar function, t_0, $x(t_0)$, t_f, $x(t_f)$ are given, and all functions are smooth. It should be mentioned here that t in the above equation is not necessarily time (although in control problems it most likely is); t simply denotes the dependent variable. The function x then which minimizes J satisfies the so–called Euler–Lagrange equations,

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \ ,$$

together with the boundary conditions $x(t_0) = x_0$, $x(t_f) = x_f$.

The solutions to these equations are called the extremals. The equations are usually referred to as Euler's equations in calculus of variations textbooks and Lagrange's equations in dynamics, where L is called the Lagrangian and is the kinetic minus the potential energy of a conservative system. Again in dynamics, the fact that the Lagrangian L is a stationary value for J is called Hamilton's principle.

The Euler–Lagrange (E–L) equations are in general 2nd order nonlinear differential equations, which means that we need two boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_f$ to solve them. Existence, however, is not guaranteed here. This is not a Cauchy initial value problem, it is called a two–point boundary value problem (more later) and can be rather difficult to solve numerically.

Some particular cases of E–L are:

1. Suppose that $L(x, \dot{x}, t)$ is independent of x (this is called an ignorable coordinate in dynamics). Then, E–L results in

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \Longrightarrow \frac{\partial L}{\partial \dot{x}} = \text{const.}$$

   which is the principle of conservation of conjugate momentum in dynamics.

2. Suppose we have a time invariant system and $L(x, \dot{x}, t)$ is independent of t. Then,

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial x \partial \dot{x}}\dot{x} - \frac{\partial^2 L}{\partial \dot{x}^2}\ddot{x} = 0 \ ,$$

   or

$$\dot{x}\frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial x \partial \dot{x}}\dot{x}^2 - \frac{\partial^2 L}{\partial \dot{x}^2}\ddot{x}\dot{x} = \frac{d}{dt}\left[L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right] = 0 \ .$$

   This of course means that

$$L - \dot{x}\frac{\partial L}{\partial \dot{x}} = \text{const.} \ ,$$

   which is the conservation of Hamiltonian.

3. If $L(x, \dot{x}, t)$ is independent of $\dot{x}$, then E–L becomes simply $\frac{\partial L}{\partial x} = 0$.

6.4 Example: The Brachystochrone Problem

The brachystochrone problem is one of the oldest problems that in fact initiated efforts towards calculus of variations. It can be simply stated as follows: Given a point O in a vertical plane with coordinates (t_0, x_0) and another point also in the same vertical plane with coordinates (t_f, x_f) find the shape of a curve connecting the two points such that a frictinless mass can start at O with zero speed and slide down in minimal time. The geometry is shown in Figure 28. We should exercise caution here in that t is not time; x and t are the two spatial coordinates of the problem.

To formulate the problem we use the kinetic energy $\frac{mv^2}{2}$ and the potential energy $-mgx$. Conservation of energy requires that $mv^2 - 2mgx = 0$ from which $v = \sqrt{2gx}$. The elapsed

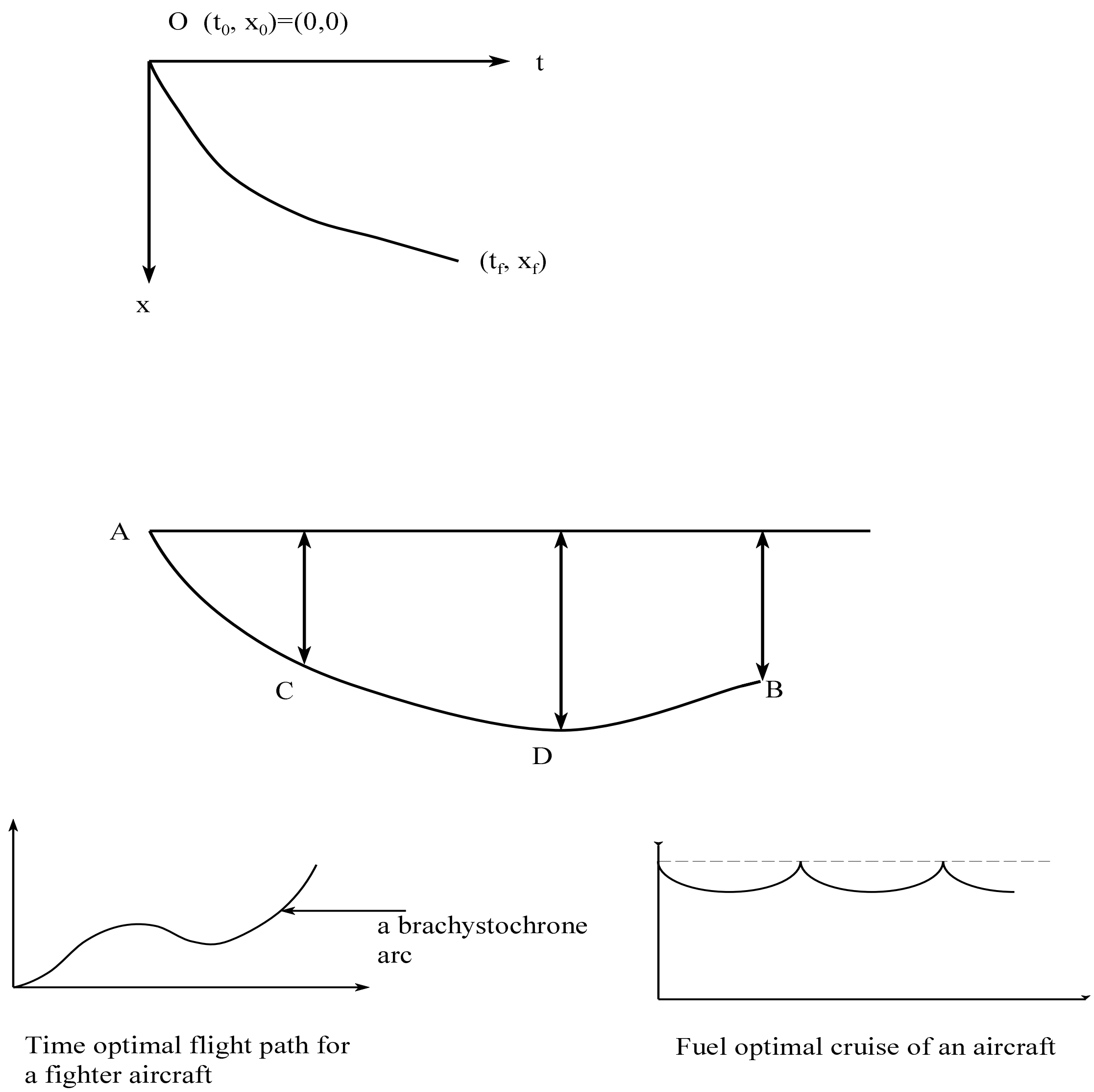

Figure 1: The brachystochrone problem

time is,

$$d\tau = \frac{ds}{v} = \frac{\sqrt{dx^2 + dt^2}}{\sqrt{2gx}} = \frac{\sqrt{1+\dot{x}^2}}{\sqrt{2gx}} dt \ .$$

The total elapsed time to minimize is then given by,

$$T = \int d\tau = \frac{1}{\sqrt{2g}} \int_{t_0=0}^{t_f} \sqrt{\frac{1+\dot{x}^2}{x}} dt \ .$$

Since the Lagrangian

$$L(x, \dot{x}, t) = \sqrt{\frac{1+\dot{x}^2}{x}} \ ,$$

is independent of t, the E–L equations become

$$\begin{aligned}
&L - \dot{x}\frac{\partial L}{\partial \dot{x}} = \text{const.} \Rightarrow \frac{\sqrt{1\dot{x}^2}}{\sqrt{x}} - \frac{\dot{x}2\dot{x}}{2\sqrt{x}\sqrt{1+\dot{x}^2}} = C \Rightarrow \\
&1 + \dot{x}^2 - \dot{x}^2 = C\sqrt{x(1+\dot{x}^2)} \Rightarrow x(1+\dot{x}^2) = C_1 \Rightarrow \\
&tx = \sqrt{\frac{C_1 - x}{x}} \Rightarrow dt = \sqrt{\frac{x}{C_1 - x}} dx \ .
\end{aligned}$$

If we let

$$x = C_1 \sin^2\theta \ ,$$

we get

$$dx = 2C_1 \sin\theta \cos\theta \, d\theta \ ,$$

and

$$dt = 2C_1 \sin^2\theta \, d\theta = C_1(1 - \cos 2\theta) \, d\theta \ .$$

Integrating,

$$t = C_1\left(\theta - \frac{\sin 2\theta}{2}\right) + C_2 \ .$$

Since $x(\theta = 0) = 0$ and $t(\theta = 0) = 0$ we get,

$$\begin{aligned}
x &= \frac{C_1}{2}(1 - \cos 2\theta) \ , \\
t &= \frac{C_1}{2}(2\theta - \sin 2\theta) \ .
\end{aligned}$$

Geometrically, these equations represent (parametrically) an arc of a cycloid generated by rotating a circle of radius $C_1/2$ by an angle 2θ. The two constants C_1 and θ can be determined by enforcing the remaining two boundary conditions,

$$x(\theta_f) = x_f \quad \text{and} \quad t(\theta_f) = t_f \ .$$

Some comments on the brachystochrone are:

1. Every sub–arc of a brachystochrone with appropriate boundary velocities is by itself a brachystochrone. With regards to Figure 28, if A–B is a brachystochrone with $v_A = 0$ and $v_B = \sqrt{2gh_B}$, then the brachystochrone between points C and D with velocities $v_C = \sqrt{2gh_C}$ and $v_D = \sqrt{2gh_D}$ is precisely the arc C–D. This is called the *Principle of Optimality.*

2. A brachystochrone remains optimal after time reversal.

3. The brachystochrone helps make "strange" results in optimal control look more plausible, see Figure 28 for a couple of possible examples.

6.5 Optimality Conditions

We can use calculus of variations to derive the optimal control. We seek a function of time $u(t)$ to minimize J subject to the state equations $\dot{x} = f(x, u)$. Ordinary calculus can be used to solve for a parameter to minimize a scalar. Calculus of variations is used to solve for a function to minimize a scalar J. This is similar to the previous E–L equations, except that here we need to satisfy the state equations as well. The approach is directly parallel to the Lagrange multiplier method for parameter optimization subject to a constraint.

The final result is as follows: In order to solve

$$\begin{aligned} &\min J = K(x_0, x_f) + \int_{t_0}^{t_f} L(x, u)\, dt\ , \\ &\text{such that } \dot{x} = f(x, u)\ , \end{aligned}$$

we define the Hamiltonian

$$H(x, p, u) = p^T f(x, u) - L(x, u)\ ,$$

where x is the state vector, and p is an unknown vector (called the co–state vector). The necessary conditions for optimality are the following sets of equations:

1. The state equations,
$$\dot{x} = \frac{\partial H}{\partial p} = f(x, u)\ .$$

2. The adjoint equations,
$$\dot{p} = -\frac{\partial H}{\partial x}\ .$$

3. Maximization of Hamiltonian,
$$\frac{\partial H}{\partial u} = 0\ ,$$
which is known as Pontryagin's maximum principle.

4. Boundary conditions,
$$\delta K + \left[p^T \delta x - H \delta t\right]_{t_0}^{t_f} = 0\ .$$

Solution of these formidable equations yields the optimal control law u. This is a very difficult task, and even when it is possible, usually the procedure yields an open loop control; i.e., u is obtained as a function of time rather than state. A special case where solution can be obtained in closed loop form is the Linear Quadratic Regulator (LQR) problem.

6.6 The Linear Quadratic Regulator

Suppose we have a linear system,

$$\dot{x} = Ax + Bu \ ,$$

and a quadratic cost function,

$$J = \tfrac{1}{2} x_f^T F x_f + \tfrac{1}{2} \int_{t_0}^{t_f} \left[x^T Q x + u^T R u \right] \, dt \ ,$$

where x_0, t_0, t_f are given (fixed) and x_f is free to vary. This is the LQR problem: we seek a control law u to minimize J. It should be emphasized that the above matrices A, B, Q, R are assumed, in general, to be functions of time. This is our first attempt, so far, to design a control law for a linear, time–varying system.

The weighting matrices F, Q, R are symmetric and positive definite and are at the discretion of the designer. Q is the state weighting matrix, R penalizes the control effort, and F penalizes the final state (or miss distance). Relatively small elements of Q compared to R will result in a control law which will tolerate errors in x with low control effort u. On the other hand, if Q is made large compared to R this will result in tight control; small errors in the state with considerable control effort. We can also use different values of the entries of Q (or R). For example, say the $(2,2)$ element of Q is large compared to the rest. This will result in improved control of the state x_2 at the expense of control accuracy of the other states and more control effort.

In order to solve the LQR problem we apply the general equations of optimal control. The Hamiltonian is

$$H(x, p, u) = p^T (Ax + Bu) - \tfrac{1}{2} (x^T Q x + u^T R u) \ ,$$

and the necessary conditions for optimality are

$$\begin{aligned}
&\dot{x} = \frac{\partial H}{\partial p} \Longrightarrow \dot{x} = Ax + Bu \ , \\
&\dot{p} = -\frac{\partial H}{\partial x} \Longrightarrow \dot{p} = -A^T p + Qx \ , \\
&\frac{\partial H}{\partial u} = 0 \Longrightarrow B^T p - Ru = 0 \Longrightarrow u = R^{-1} B^T p \ .
\end{aligned}$$

The boundary conditions are

$$\left[p^T \delta x - H \delta t \right]_{t_0}^{t_f} + \delta \left(\tfrac{1}{2} x_f^T F x_f \right) = 0 \ ,$$

or

$$p^T(t_f)\delta x_f - p^T(t_0)\delta x_0 - H(t_f)\delta t_f + H(t_0)\delta t_0 + x_f^T F \delta x_f = 0 \ .$$

Since x_0, t_0, and t_f are fixed we have

$$\delta x_0 = \delta t_0 = \delta t_f = 0 \ ,$$

and the boundary condition becomes

$$\begin{aligned} & p^T(t_f)\delta x_f + x_f^T F \delta x_f = 0 \Rightarrow \\ & \left[p^T(t_f) + x_f^T F\right] \delta x_f = 0 \ . \end{aligned}$$

Since x_f is free, its variation δx_f is arbitrary. Therefore, the quantity inside the square brackets must vanish, and this produces the desired boundary condition in the form

$$p(t_f) = -F x(t_f) \ .$$

In summary, the problem we have to solve is

$$\begin{aligned} & \dot{x} = Ax + BR^{-1}B^T p \ , \\ & \dot{p} = Qx - A^T p \ , \\ & x(t_0) = x_0 \ , \\ & p(t_f) = -F x(t_f) \ . \end{aligned}$$

Solution of these ordinary differential equations will provide $p(t)$ and this will allow calculation of u as a function of time from $u = R^{-1}B^T p(t)$. However, solving these equations is not as easy as it may seem. Notice that for a numerical integration of $\dot{x}$ and $\dot{p}$ we need to know the initial conditions at t_0; i.e., $x(t_0)$ and $p(t_0)$. But we know $p(t_f) = -F x(t_f)$ instead of $p(t_0)$. This is called a two–point boundary value problem with half of the boundary conditions at t_0 and the other half at t_f. Solution of two–point boundary value problems requires iterative (shooting) techniques: assume an initial condition $p(t_0)$, integrate numerically the system and at the end check whether the condition $p(t_f) = -F x(t_f)$ is satisfied, if it is not change the initial condition $p(t_0)$ and iterate until convergence. To make things worse, even if we could easily solve this problem, still the optimal control u would be open loop, $u(t)$ instead of $u(x)$.

Kalman's idea comes here to the rescue: Let

$$p(t) = -S(t)x(t) \ ,$$

where $S(t)$ is a symmetric positive definite matrix to be determined. Then we have

$$\begin{aligned} \dot{p} &= -\dot{S}x - S\dot{x} \\ &= -\dot{S}x - S\left(Ax + BR^{-1}B^T p\right) \ , \end{aligned}$$

or

$$\begin{aligned} & Qx - A^T p = -\dot{S}x - SAx - SBR^{-1}B^T p \Rightarrow \\ & Qx + A^T Sx = -\dot{S}x - SAx + SBR^{-1}B^T Sx \Rightarrow \\ & -\dot{S}x = \left(A^T S + SA - SBR^{-1}B^T S + Q\right) x \ , \end{aligned}$$

and since this must be true for all x we get

$$-\dot{S} = A^T S + SA - SBR^{-1}B^T S + Q \ ,$$

with

$$S(t_f) = F \ .$$

This is called a Riccati matrix differential equation. Therefore, we can obtain $S(t)$ by backwards integration of the Riccati matrix differential equation, and then obtain the closed loop optimal control law by

$$u = -R^{-1}B^T S(t)x \ ,$$

a linear state feedback with time varying gains.

For the case of constant A, B, Q, R matrices and $t_f \to \infty$, we have the steady state problem $\dot{S} = 0$. In this case the optimal closed loop control law is

$$u = -R^{-1}B^T Sx \ ,$$

where S is found by solving the algebraic Riccati equation (ARE) for the positive definite S,

$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \ .$$

This is a nonlinear algebraic equation in the elements of S and it may admit multiple solutions, only one of them is positive definite though, and this is the one that we seek. See the `lqr` command for solution of the LQR problem using MATLAB.

Recall that previously we were using pole (eigenvalue) placement to produce arbitrary closed–loop eigenvalues. Here we have a technique more suited for large, multivariable systems in which we choose the weighting matrices Q and R. The mathematics then yields a set of closed loop eigenvalues which are guaranteed to be stable (we will see why shortly) but over which we have no direct control. If the closed loop eigenvalues are not acceptable, it may be necessary to change the weighting matrices Q and R and iterate. If the errors in the state x_i are too large, it would be necessary to raise q_{ii}. If there is excessive use of control u_j, it would be necessary to raise r_{jj}. This would cause the state or control with the increased weighting in J to be reduced in the next design (iteration) at the expense of increased errors in the other states and/or increased usage of the other controls.

How do we know that the LQR design yields a stable system though? We can show stability by using Lyapunov's method. Choose

$$V(x) = x^T Sx \ ,$$

as a Lyapunov function candidate, where S is the positive definite solution of the Riccati equation. Since S is a positive definite matrix, $V(x) > 0$. Its time derivative is

$$\begin{aligned}
\dot{V}(x) &= \dot{x}Sx + x^T \dot{S} + x^T S\dot{x} \\
&= (Ax + Bu)^T Sx + x^T \dot{S} + x^T S(Ax + Bu) \\
&= x^T(\dot{S} + A^T S + SA - 2SBR^{-1}B^T S)x \\
&= x^T(-SBR^{-1}B^T S - Q)x \\
&= -x^T SBR^{-1}B^T Sx - x^T Qx \ .
\end{aligned}$$

Let $z = R^{-1}B^T S x$ be some vector, then

$$\dot{V}(x) = -z^T R z - x^T Q x < 0 \ ,$$

since Q, R are positive definite matrices. Therefore, $V(x)$ is a Lyapunov function for the LQR design, and since

$$\begin{aligned} V(x) &> 0 \quad \text{and} \\ \dot{V}(x) &< 0 \ , \end{aligned}$$

the design will always yield a stable system (as long as the Riccati equation supplies the positive definite solution matrix S).

As an example, say we have $\dot{x} = 2x + u$, a scalar system. The open loop pole is $s - 2 = 0$ or $s = 2$, so it is unstable. We wish to control x near zero and minimize

$$J = \int_0^\infty (qx^2 + ru^2)\, dt \ .$$

Suppose we want to use $q = 0.25$ and $r = 1$. Then the ARE is $2k + 2k - k \cdot 1 \cdot 1^{-1} \cdot 1 \cdot k + 0.25 = 0$, or $k^2 - 4k - 0.25 = 0$. The positive root is $k = 4.06$ and the optimal control is

$$u = -1^{-1} \cdot 1 \cdot 4.06x = -4.06x \ .$$

The closed loop eigenvalue is $\det(2 - 4.06 - s) = 0$ or $s = -2.06$, and the closed loop response is $x(t) = x(t_0)e^{-2.06t}$. If we wish to reduce the error in x faster at the expense of using more control we can raise q. If we redesign for $q = 4$, $r = 1$ we get $k = 4.83$, $u = -4.83x$, and $x(t) = x(t_0)e^{-2.83t}$. If we wish to reduce the amount of control used at the expense of slower response, we can raise r. If we redesign for $q = 0.25$ and $r = 10$, we get $k = 40.06$, $u = -4x$, and $x(t) = x(t_0)e^{-2t}$.

Example: Consider the submarine equations of motion

$$\begin{aligned} \dot{\theta} &= q \ , \\ \dot{w} &= a_{11}Uw + a_{12}Uq + a_{13}z_{GB}\theta + b_1U^2\delta \ , \\ \dot{q} &= a_{21}Uw + a_{22}Uq + a_{23}z_{GB}\theta + b_2U^2\delta \ , \\ \dot{z} &= -U\theta + w \ . \end{aligned}$$

One common logic in selecting the weighting matrices Q and R in the performance index J is to say that we are willing to use control u_{j0} when state error x_{i0} is reached. We can make Q and R diagonal with

$$\begin{aligned} q_{ii} &= \frac{1}{x_{i0}^2} \ , \quad i = 1, 2, \ldots, n \quad (n \text{ states}) \ , \\ r_{jj} &= \frac{1}{u_{j0}^2} \ , \quad j = 1, 2, \ldots, m \quad (m \text{ controls}) \ . \end{aligned}$$

In our case the performance index is, in general,

$$J = \int (q_{11}\theta^2 + q_{22}w^2 + q_{33}q^2 + q_{44}z^2 + r\delta^2)\, dt \ .$$

In this case we want to control θ and z near zero (their nominal values) and use a reasonable amount of dive planes to do the job. We assume it would be reasonable to use 5^o dive planes for depth control when the pitch angle deviates 3^o from zero or the boat reaches a depth deviation of 1.5 feet (about one tenth of the length). We, therefore, assume all terms in Q and R to be zero except,

$$
\begin{aligned}
q_{11} &= \left(\frac{3}{57.3}\right)^{-2} = 364.8 \text{ weighting on } \theta^2 \ , \\
q_{44} &= (1.5)^{-2} = 0.444 \text{ weighting on } z^2 \ , \\
r_{11} &= \left(\frac{5}{57.3}\right)^{-2} = 133.3 \text{ weighting on } \delta^2 \ .
\end{aligned}
$$

The performance index is

$$J = \int \left(q_{11}\theta^2 + q_{44}z^2 + r_{11}\delta^2\right)\, dt \ ,$$

and the control law then becomes

$$\delta = -(-2.75570\theta - 0.5457w - 2.7657q + 0.0577z) \ ,$$

and the closed loop poles are

$$-0.5207 \pm 0.2841i \quad \text{and} \quad -0.1197 \pm 0.0704i \ .$$

A numerical simulation in terms of z and δ is shown in Figure 29 by the solid curves. If we decide to use 5^o dive planes for depth control when the pitch angle deviates 3^o from zero or the boat reaches a depth deviation 0.5 feet from zero, we expect a tighter control law: the same dive plane angle is commanded for one third the error in z. In this case the control law is

$$\delta = -(-4.61870\theta - 0.5177w - 4.5379q + 0.1732z) \ ,$$

and the closed loop poles are

$$-0.4901 \pm 0.2819i \quad \text{and} \quad -0.2267 \pm 0.1111i \ .$$

The dominant pole is more negative in this case, as it should. The results of this simulation are also shown in Figure 29 with the dotted curves, the response is faster at the expense of more plane activity.

Other performance indices are also possible. Suppose the objective is to keep the submarine at constant depth, $z = 0$, while minimizing the added drag due to dive plane activity. The design is then for a depth controller which will minimize the added drag on the boat due to its deviations from the equilibrium (nominal) level flight path $x = [\theta, w, q, z]^T = [0, 0, 0, 0]^T$ and control $\delta = 0$. To formulate the problem we need the longitudinal (surge) equation of motion, which is (see ME 4823 for details)

$$(m - X_{\dot{u}})\dot{u} = X_{qq}q^2 + (X_{wq} - m)wq + X_{ww}w^2 + X_{UU}U^2 + X_{\delta\delta}\delta^2 + T_{\text{prop}} \ ,$$

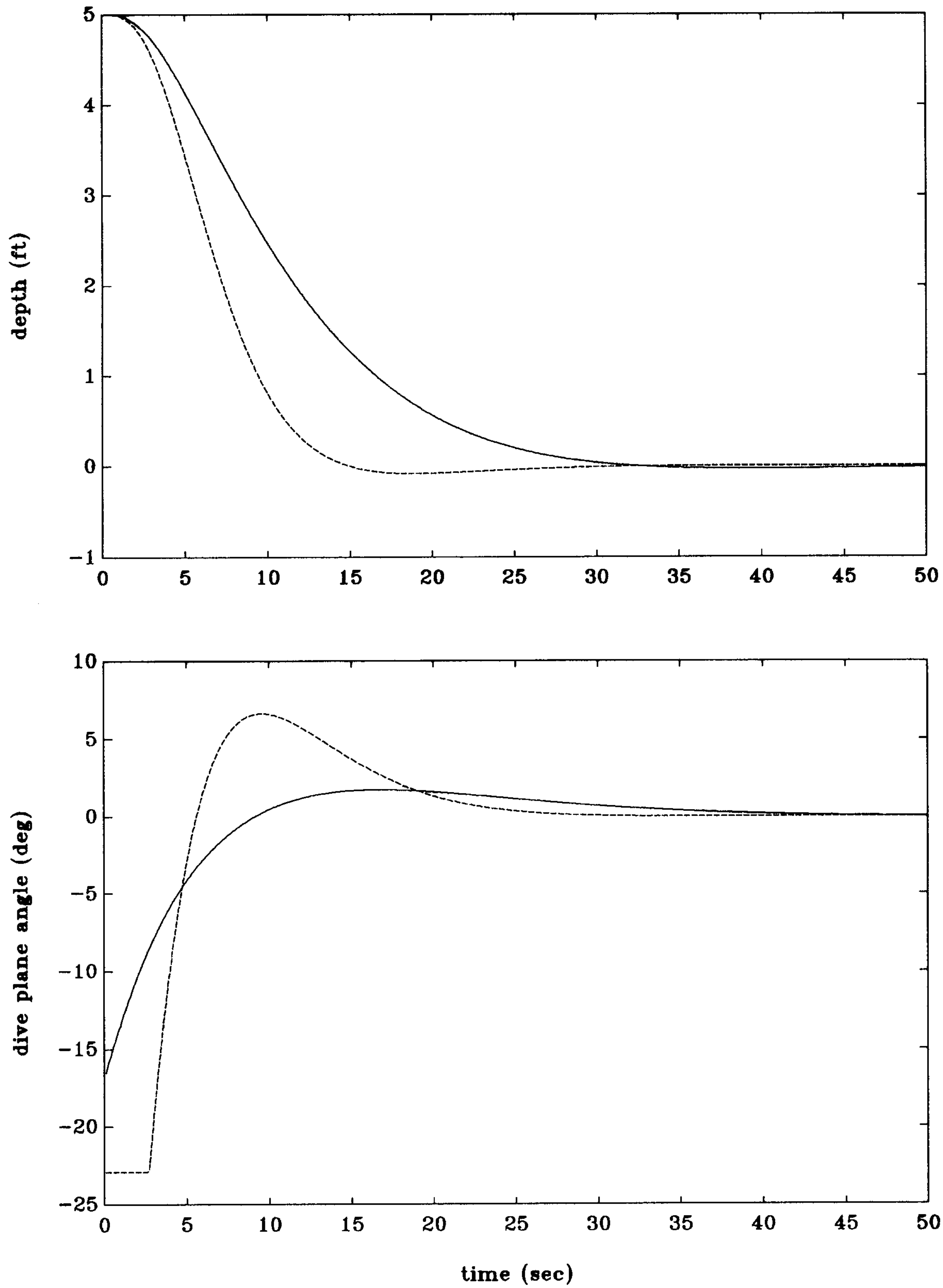

Figure 2: LQR simulation for a slow (solid) and a tight control (dotted) law

where X_{UU} represents the drag coefficient in straight line motion, T_{prop} is the propulsive force, and the terms X_{qq}, X_{wq}, X_{ww}, $X_{\delta\delta}$ produce the added drag due to nonzero w, q, δ. The control objectives here are:

$$\begin{aligned} \text{depth control} &: \text{ minimize } z^2, \text{deviation from desired ,} \\ \text{added drag} &: \text{ minimize } -F_d \text{ ,} \end{aligned}$$

where

$$-F_d = -X_{qq}q^2 - (X_{wq} - m)wq - X_{ww}w^2 - X_{\delta\delta}\delta^2 \ .$$

The weighting index is then

$$J = \int (q_{44}z^2 - F_d)\, dt \ ,$$

or

$$J = \int (-X_{qq}q^2 - (X_{wq} - m)wq - X_{ww}w^2 - X_{\delta\delta}\delta^2)\, dt \ .$$

Therefore, we can use

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -X_{ww} & -\frac{1}{2}(X_{wq} - m) & 0 \\ 0 & -\frac{1}{2}(X_{wq} - m) & -X_{qq} & 0 \\ 0 & 0 & 0 & q_{44} \end{bmatrix} ,$$

and

$$R = \begin{bmatrix} -X_{\delta\delta} \end{bmatrix} ,$$

where q_{44} is the weighting factor between minimizing depth deviations and minimizing drag. Relatively large values of q_{44} will penalize depth deviations heavily and will result in tight control with increased plane activity (this may be required in operations at periscope depth, for example). On the other hand, if q_{44} is chosen small, the resulting control law will penalize control activity more resulting in minimizing drag and fuel efficiency, with larger depth deviations from nominal.

6.7 Time Optimal Control of a Double Integral Plant

Consider the dynamical system,

$$M\ddot{x} = F \ .$$

If we define,

$$x_1 = x \ , \quad x_2 = \dot{x} \ , \quad u = \frac{F}{M} \ ,$$

we can write it in state space form as,

$$\begin{aligned} \dot{x}_1 &= x_2 \ , \\ \dot{x}_2 &= u \ . \end{aligned}$$

We also assume the control constraints

$$|u| \leq 1 \ ,$$

and the initial conditions,

$$x_1(0) = x_{10}\ , \quad x_2(0) = x_{20}\ , \quad x_1(T) = x_2(T) = 0\ .$$

We want to minimize the time to fly,

$$\min J = \int_0^T dt\ .$$

The Hamiltonian is

$$H(x,p,u,t) = p^T f(x,u,t) - L(x,u,t) = p_1 x_2 + p_2 u - 1\ .$$

The necessary conditions for optimality are

$$\begin{aligned}
\dot{x}_1 &= \frac{\partial H}{\partial p_1} = x_2\ , \\
\dot{x}_2 &= \frac{\partial H}{\partial p_2} = u\ , \\
\dot{p}_1 &= -\frac{\partial H}{\partial x_1} = 0\ , \\
\dot{p}_2 &= -\frac{\partial H}{\partial x_2} = -p_1\ .
\end{aligned}$$

Pontryagin's maximum principle states that u must maximize $H = p_1 x_2 + p_2 u - 1$. Therefore, the optimal control needs to maximize $p_2 u$ (since the rest of H does not depend on u). We can see that if p_2 is positive, u must get the maximum positive value (in this case $+1$), while if p_2 is negative, u must be -1. Therefore, the optimal control is given by

$$u = \mathrm{sgn}[p_2(t)] = \ +1 \text{ if } p_2 > 0 \text{ and } -1 \text{ if } p_2 < 0\ .$$

The optimal trajectory is given by the solution to,

$$\begin{aligned}
&\dot{x} - 1 = x_2\ , \\
&\dot{x}_2 = \mathrm{sgn}(p_2)\ , \\
&\dot{p}_1 = 0\ , \\
&\dot{p}_2 = -p_1\ , \\
&x_1(0) = x_{10},\ x_2(0) = x_{20},\ x_1(T) = 0,\ x_2(T) = 0\ .
\end{aligned}$$

This is a reduced system of equations, since u is eliminated by maximizing H.

To solve this system we observe that since $\dot{p}_1 = 0$ we have that $p_1 = \text{const.}$ and this means that p_2 is a first–order polynomial in t. Therefore, it can only go from positive to negative at most once in its life, which means that there are only four possible control sequences,

$$\{+1\}\ , \quad \{-1\}\ , \quad \{+1,-1\}\ , \quad \{-1,+1\}\ .$$

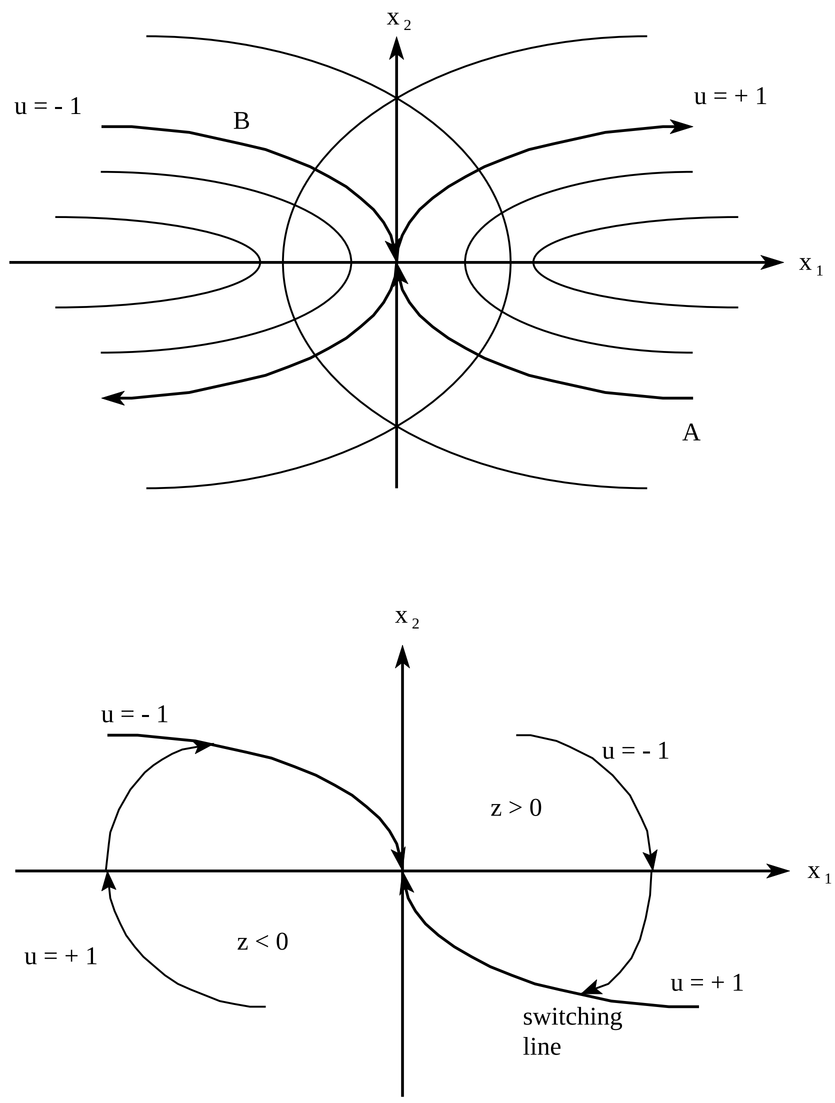

Figure 3: Time optimal control of a double integrator plant

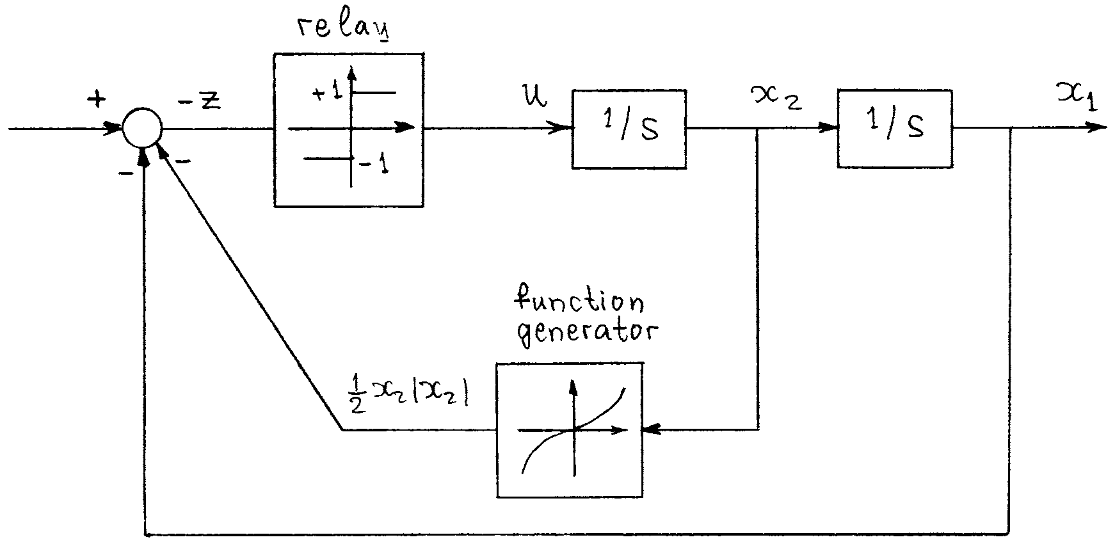

Figure 4: Time optimal control of a double integrator plant: Feedback implementation

If we let $U = \pm 1$ be the control, we have

$$\begin{aligned} x_1 &= x_{10} + x_{20}t + \frac{1}{2}Ut^2 \ , \\ x_2 &= x_{20} + Ut \ . \end{aligned}$$

If we eliminate t we can get

$$x_1 - \left(x_{10} - \frac{1}{2}Ux_{20}^2 \right) = \frac{1}{2}Ux_2^2 \ ,$$

which represents a family of parabolas as shown in Figure 30. If $u = +1$ we are located on branch A while if $u = -1$ we are on branch B. The branch that goes through the origin is called the *switching line* and it is given by

$$x_1 = -\frac{1}{2}x_2|x_2| \ .$$

To see how this optimal control works, suppose we start from an initial condition with both x_1 and x_2 positive. We apply control $u = -1$ until we hit the switching line, there we switch to $u = +1$ and we land at the origin with zero velocity.

A feedback control implementation is shown in Figure 31. We define

$$z = x_1 + \frac{1}{2}x_2|x_2| \ ,$$

which means that the switching line is $z = 0$. Therefore, we get the optimal control through a switch $u = -1$ when $z > 0$ and $u = +1$ when $z < 0$. We should point out that in this case the final portion of the state trajectory follows the switching curve, this is not typical for all systems though. Since the optimal control switches from positive to negative we call it *bang–bang* control. Most minimum time control problems lead to bang–bang controllers. Pontryagin has shown that for a system of order n with negative real poles and scalar u, $|u| \leq 1$, the optimal control switches at most $n - 1$ times.